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INEQUALITIES FOR B-CONVOLUTION OPERATORS

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Abstract. The B-convolution operators generated by the generalized shift operators associated with the Laplace-Bessel operator are considered and three inequalities are proved for these operators. The first two inequalities are O'Neil type inequalities and third inequality is a generalization of the Young inequality for the B-convolution integrals. The last inequality is also an extension of the Hardy-Littlewood-Sobolev theorem for the B-fractional integrals.

Keywords: Laplace-Bessel differential operator, B-convolution, O'Neil inequality, Young inequality, Lorentz space, B-fractional integral.

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1. Introduction and main results

It is well known (see, for example, [5]) that the generalized shift operator

$$T^y f(x) = C_{k,\gamma} \int\limits_0^\pi \dots \int\limits_0^\pi f\left((x',y')_\alpha, x''-y''\right) d\nu(\alpha),$$

where $C_{k,\gamma} = \pi^{-\frac{k}{2}} \prod\limits_{i=1}^{k} \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}$, $(x'y')_\alpha = ((x_1,y_1)_{\alpha_1}, \dots, (x_k,y_k)_{\alpha_k})$, $(x_i,y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2}$, $1 \le i \le k$, $(x',x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $1 \le k \le n$, $d\nu(\alpha) = \prod\limits_{i=1}^{k} \sin^{\gamma_i - 1} \alpha_i d\alpha_i$, is closely related to the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{k} B_i + \sum\limits_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}$, where $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ and T^y generates the corresponding B-convolution

$$(f \otimes \varphi)_\gamma(x) = \int\limits_{\mathbb{R}^n_{k,+}} f(y) \left(T^y \varphi(x)\right) (y')^\gamma dy,$$

where $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index, $(y')^\gamma = y_1^{\gamma_1} \cdots y_k^{\gamma_k}$ and $\mathbb{R}^n_{k,+} = \{x = (x_1, \dots, x_n),\ x_1 > 0, \dots, x_k > 0\}$.

There are a lot of papers that studied B-convolution operators and related topics associated with the Laplace-Bessel differential operator (see, for example [1, 3, 4, 5, 6, 7]).

In the present paper some inequalities for B-convolution operators are proved.

The following theorem was proved in [4].

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Theorem 1.1. *Let f and φ be measurable functions on $\mathbb{R}^n$, then for all $t > 0$ the following inequality holds*

$$(f * \varphi)^{**}(t) \le t f^{**}(t)\varphi^{**}(t) + \int_t^\infty f^*(s)\varphi^*(s)ds, \tag{1}$$

where $f * \varphi(x) = \int\limits_{\mathbb{R}^n} f(x-y)\,\varphi(y)\,dy$.

The inequality (1) is known as the O'Neil inequality. In [4], the O'Neil type inequality for the B-convolution operators was obtained in the following form.

Theorem 1.2. [4] *Let f and φ be measurable functions on $\mathbb{R}^n_{k,+}$, then for all $t > 0$ the following inequality holds*

$$(f \otimes \varphi)^{**}_\gamma(t) \le C_{k,\gamma}\left(f^{**}_\gamma(t) \int_0^t \varphi^{**}_\gamma(s)ds + \int_t^\infty f^*_\gamma(s)\varphi^{**}_\gamma(s)ds \right). \tag{2}$$

A simple comparison shows that the inequality (2) is not an exact analogue of the inequality (1). Moreover, since φ^{**}_γ is non-increasing on $(0,\infty)$ and $\varphi^*_\gamma \le \varphi^{**}_\gamma$ we have $f^{**}_\gamma(t)\int\limits_0^t \varphi^{**}_\gamma(s)ds \ge$ $\ge t f^{**}_\gamma(t)\varphi^{**}_\gamma(t)$ and $\int\limits_t^\infty f^*_\gamma(s)\varphi^{**}_\gamma(s)ds \ge \int\limits_t^\infty f^*_\gamma(s)\varphi^*_\gamma(s)ds$.

The first main result of our paper is the following theorem which gives an exact analogue of the O'Neil inequality for the B-convolution.

Theorem 1.3. *Let f and φ be measurable functions on $\mathbb{R}^n_{k,+}$, then for all $t > 0$ the following inequality holds*

$$(f \otimes \varphi)^{**}_\gamma(t) \le t f^{**}_\gamma(t)\varphi^{**}_\gamma(t) + \int_t^\infty f^*_\gamma(s)\varphi^*_\gamma(s)ds. \tag{3}$$

In the following theorem we give another kind of estimate of the maximal function of the rearrangement of the B-convolution operator.

Theorem 1.4. *Let f and φ be measurable functions on $\mathbb{R}^n_{k,+}$, then for all $t > 0$ the following inequality holds*

$$(f \otimes \varphi)^{**}_\gamma(t) \le \int_t^\infty f^{**}_\gamma(s)\varphi^{**}_\gamma(s)ds. \tag{4}$$

In Theorem 1.5 we obtain the generalization of the Young inequality for the B-convolution operators.

Theorem 1.5. *If $f \in L_{p_1,q_1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\varphi \in L_{p_2,q_2,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $\dfrac{1}{p_1} + \dfrac{1}{p_2} > 1$, then $(f \otimes \varphi) \in L_{p_0,q_0,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ where $\dfrac{1}{p_1} + \dfrac{1}{p_2} - 1 = \dfrac{1}{p_0}$ and $q_0 \ge 1$ is any number such that $\dfrac{1}{q_1} + \dfrac{1}{q_2} \ge \dfrac{1}{q_0}$. Moreover,*

$$\|(f \otimes \varphi)\|_{p_0,q_0,\gamma} \le 3p_0 \|f\|_{p_1,q_1,\gamma}\|\varphi\|_{p_2,q_2,\gamma}. \tag{5}$$

As an application, we check that the Hardy-Littlewood-Sobolev theorem for the B-fractional integrals ([4]) is a particular case of Theorem 1.5.

2. Preliminaries

For $1 \leq p \leq \infty$, the Lebesgue space $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is defined as

$$L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right) = \{f : f \text{ is measurable on } \mathbb{R}^n_{k,+}\,, \|f\|_{p,\gamma} < \infty\},$$

where $\|f\|_{p,\gamma}$ is defined by

$$\|f\|_{p,\gamma} = \begin{cases} \left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \operatorname*{ess\,sup}\limits_{x \in \mathbb{R}^n_{k,+}} |f(x)|, & \text{if } p = \infty. \end{cases}$$

Let $1 \leq p \leq \infty$. If f is in $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and φ is in $L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, then the function $f \otimes \varphi$ belongs to $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and

$$\|f \otimes \varphi\|_{p,\gamma} \leq \|f\|_{p,\gamma}\|\varphi\|_{1,\gamma}.$$

For any measurable set $E \in \mathbb{R}^n_{k,+}$, let $|E|_\gamma = \int\limits_E (x')^\gamma dx$. Suppose f is a measurable function defined on $\mathbb{R}^n_{k,+}$. The distribution function $f_{*,\gamma}$ of the function f is given by

$$f_{*,\gamma}(s) = |\{x : x \in \mathbb{R}^n_{k,+}, |f(x)| > s\}|_\gamma, \text{ for } s \geq 0.$$

The distribution function $f_{*,\gamma}$ is non-negative, non-increasing and continuous from the right. With the distribution function we associate the non-increasing rearrangement of f on $[0,\infty)$ defined by

$$f^*_\gamma(t) = \inf\{s > 0 : f_{*,\gamma}(s) \leq t\}.$$

Some elementary properties of $f_{*,\gamma}$ and f^*_γ are listed below. The proofs of them can be found in [2].

(a) If $f_{*,\gamma}$ is continuous and strictly decreasing, then f^*_γ is the inverse of $f_{*,\gamma}$, that is $f^*_\gamma = (f_{*,\gamma})^{-1}$.

(b) f^*_γ is continuous from the right.

(c) $m_{f^*_\gamma}(s) = f_{*,\gamma}(s)$, for all $s > 0$, where $m_{f^*_\gamma}$ is a distribution function of the function f^*_γ with respect to Lebesgue measure m on $(0,\infty)$, that is $m_{f^*_\gamma}(s) = \int\limits_{f^*_\gamma(t) > s} dt$.

(d)

$$\int_0^t f^*_\gamma(s)ds = tf^*_\gamma(t) + \int_{f^*_\gamma(t)}^\infty f_{*,\gamma}(s)ds. \tag{6}$$

(e) If $f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 \leq p < \infty$, then

$$\left(\int\limits_{\mathbb{R}^n_{k,+}} |f(x)|^p \left(x'\right)^\gamma dx(x)\right)^{\frac{1}{p}} = \left(p\int_0^\infty s^{p-1} f_{*,\gamma}(s)ds\right)^{\frac{1}{p}} = \left(\int_0^\infty \left(f^*_\gamma(t)\right)^p dt\right)^{\frac{1}{p}}.$$

Furthermore, in the case $p = \infty$,

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| = \inf\{s : f_{*,\gamma}(s) = 0\} = f^*_\gamma(0),$$

f_γ^{**} will denote the maximal function of f_γ^* defined by

$$f_\gamma^{**}(t) = \frac{1}{t}\int_0^t f_\gamma^*(u)du, \text{ for } t > 0.$$

Note the following properties of f_γ^{**}:

(i) f_γ^{**} is nonnegative, non-increasing and continuous on $(0,\infty)$ and $f_\gamma^* \leq f_\gamma^{**}$.

(ii) $(f+g)_\gamma^{**} \leq f_\gamma^{**} + g_\gamma^{**}$.

(iii) If $|f_n| \uparrow |f|$ a.e., then $(f_n)_\gamma^{**} \uparrow f_\gamma^{**}$.

For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L_{p,q,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is defined as

$$L_{p,q,\gamma}\left(\mathbb{R}^n_{k,+}\right) = \{f : f \text{ is measurable on } \mathbb{R}^n_{k,+}, \|f\|_{p,q,\gamma} < \infty\},$$

where $\|f\|_{p,q,\gamma}$ is defined by

$$\|f\|_{p,q,\gamma} = \begin{cases} \left(\int\limits_0^\infty \left(t^{\frac{1}{p}} f_\gamma^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1 \leq p < \infty, 1 \leq q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_\gamma^{**}(t), & 1 \leq p \leq \infty, q = \infty. \end{cases}$$

Note that if $1 < p \leq \infty$ then $L_{p,p,\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Moreover,

$$\|f\|_{p,\gamma} \leq \|f\|_{p,p,\gamma} \leq p'\|f\|_{p,\gamma}, \tag{7}$$

where $p' = \begin{cases} \dfrac{p}{p-1}, & 1 < p < \infty, \\ 1, & p = \infty. \end{cases}$

For $p > 1$, the space $L_{p,\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ is known as the Marcinkiewicz space or as Weak $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Note also that $L_{1,\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right) = L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$.

If $1 < p < \infty$ and $1 < q < r < \infty$, then

$$L_{p,q,\gamma}\left(\mathbb{R}^n_{k,+}\right) \subset L_{p,r,\gamma}\left(\mathbb{R}^n_{k,+}\right).$$

Moreover,

$$\|f\|_{p,r,\gamma} \leq \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{p,q,\gamma}. \tag{8}$$

3. Proofs

Lemma 3.1. *Let f and φ be measurable functions on $\mathbb{R}^n_{k,+}$, where $\sup\limits_{x\in\mathbb{R}^n_{k,+}} |f(x)| \leq \beta$ and f vanishes outside of a measurable set E with $|E|_\gamma = r$. Then, for $t > 0$,*

$$(f \otimes \varphi)_\gamma^{**}(t) \leq \beta r \varphi_\gamma^{**}(r) \tag{9}$$

and

$$(f \otimes \varphi)_\gamma^{**}(t) \leq \beta r \varphi_\gamma^{**}(t). \tag{10}$$

Proof. Without loss of generality we can assume that the functions f and φ are nonnegative. Let $h = f \otimes \varphi$. For $a > 0$, define

$$\varphi_a(x) = \begin{cases} \varphi(x), & \text{if } \varphi(x) \leq a \\ a, & \text{if } \varphi(x) > a, \end{cases}$$

$$\varphi^a(x) = \varphi(x) - \varphi_a(x).$$

Define functions h_1 and h_2 by

$$h = f \otimes \varphi_a + f \otimes \varphi^a = h_1 + h_2.$$

Then

$$\sup_{x\in\mathbb{R}^n_{k,+}} h_2(x) \leq \sup_{x\in\mathbb{R}^n_{k,+}} f(x)\|\varphi^a\|_{1,\gamma} \leq \beta \int\limits_a^\infty \varphi_{*,\gamma}(s)ds, \tag{11}$$

because $\varphi^a(x) = 0$ whenever $\varphi(x) \leq a$. Also

$$\sup_{x\in\mathbb{R}^n_{k,+}} h_1(x) \leq \|f\|_{1,\gamma} \sup_{x\in\mathbb{R}^n_{k,+}} \varphi_a(x) \leq \beta r a \tag{12}$$

and

$$\|h_2\|_{1,\gamma} \leq \|f\|_{1,\gamma}\|\varphi^a\|_{1,\gamma} \leq \beta r \int\limits_a^\infty \varphi_{*,\gamma}(s)ds. \tag{13}$$

Now setting $a = \varphi^*_\gamma(r)$ in (11) we (12) and obtain

$$h^{**}_\gamma(t) = \frac{1}{t}\int\limits_0^t h^*_\gamma(s)ds \leq \|h\|_{\infty,\gamma} \leq \|h_1\|_{\infty,\gamma} + \|h_2\|_{\infty,\gamma} \leq$$

$$\leq \beta r \varphi^*_\gamma(r) + \beta \int\limits_{\varphi^*_\gamma(r)}^\infty \varphi_{*,\gamma}(s)ds.$$

Then using (6) we have the inequality (9).

Let us prove the inequality (10). For this purpose set $a = \varphi^*_\gamma(t)$, use (12) and (13). Then

$$th^{**}_\gamma(t) = \int\limits_0^t h^*_\gamma(s)ds \leq \int\limits_0^t (h_1)^*_\gamma(s)ds + \int\limits_0^t (h_2)^*_\gamma(s)ds \leq$$

$$\leq t\|h_1\|_{\infty,\gamma} + \int\limits_0^\infty (h_2)^*_\gamma(s)ds = t\|h_1\|_{\infty,\gamma} + \|h_2\|_{1,\gamma} \leq$$

$$\leq t\beta r \varphi^*_\gamma(t) + \beta r \int\limits_{\varphi^*_\gamma(t)}^\infty \varphi_{*,\gamma}(s)ds =$$

$$= \beta r \left(t\varphi^*_\gamma(t) + \int\limits_{\varphi^*(t)}^\infty \varphi_{*,\gamma}(s)ds \right) = \beta r t \varphi^{**}_\gamma(t).$$

Proof of Theorem 1.3. Without loss of generality we can assume that the functions f and φ are nonnegative. Let $h = f \otimes \varphi$ and fix $t > 0$. Select a nondecreasing sequence $\{s_n\}^{+\infty}_{-\infty}$ such that $s_0 = f^*_\gamma(t)$, $\lim\limits_{n\to+\infty} s_n = +\infty$, $\lim\limits_{n\to-\infty} s_n = 0$.

Let also

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n(x)$$

where

$$f_n(x) = \begin{cases} 0, & \text{if } f(x) \leq s_{n-1} \\ f(x) - s_{n-1} & \text{if } s_{n-1} < f(x) \leq s_n \\ s_n - s_{n-1} & \text{if } s_n < f(x), \end{cases}$$

Since the series converges absolutely we have

$$h=\left(\sum_{n=-\infty}^{+\infty} f_n\right)\otimes\varphi=\sum_{n=-\infty}^{+\infty}\left(f_n\otimes\varphi\right).$$

Define functions h_1 and h_2 by

$$h=\sum_{n=1}^{+\infty}\left(f_n\otimes\varphi\right)+\sum_{n=-\infty}^{0}\left(f_n\otimes\varphi\right)=h_1+h_2.$$

Estimate $(h_1)_\gamma^{**}(t)$. For this purpose use the inequality (10) with $E=\{x: f(x)>s_{n-1}\}$ and $\beta=s_n-s_{n-1}$. We have

$$(h_1)_\gamma^{**}(t)\le\sum_{n=1}^{+\infty}\left((f_n\otimes\varphi)_\gamma^{**}\right)\le$$

$$\le\sum_{n=1}^{+\infty}(s_n-s_{n-1})f_{*,\gamma}(s_{n-1})\varphi_\gamma^{**}(t)=$$

$$=\varphi_\gamma^{**}(t)\sum_{n=1}^{+\infty}f_{*,\gamma}(s_{n-1})(s_n-s_{n-1}).$$

Hence

$$(h_1)_\gamma^{**}(t)\le\varphi_\gamma^{**}(t)\int\limits_{f^*(t)}^{\infty}f_{*,\gamma}(s)ds. \tag{14}$$

To estimate $(h_2)_\gamma^{**}(t)$ we use the inequality (9). Then we can write

$$(h_2)_\gamma^{**}(t)\le\sum_{n=-\infty}^{0}\left((f_n\otimes\varphi)_\gamma^{**}\right)\le$$

$$\le\sum_{n=1}^{+\infty}(s_n-s_{n-1})f_{*,\gamma}(s_{n-1})\varphi_\gamma^{**}(f_{*,\gamma}(s_{n-1}))=$$

$$=\sum_{n=1}^{+\infty}f_{*,\gamma}(s_{n-1})\varphi_\gamma^{**}(f_{*,\gamma}(s_{n-1}))(s_n-s_{n-1}).$$

This implies that

$$(h_2)_\gamma^{**}(t)\le\int\limits_0^{f_\gamma^*(t)}f_{*,\gamma}(s)\varphi_\gamma^{**}(f_{*,\gamma}(s))ds. \tag{15}$$

We will estimate the integral on the right-hand side of (15) by making the substitution $s=f_\gamma^*(\xi)$ and integrating by parts. In order to justify the change of variable in the integral, consider a Riemann sum

$$\sum_{n=1}^{+\infty}f_{*,\gamma}(s_{n-1})\varphi_\gamma^{**}(f_{*,\gamma}(s_{n-1}))(s_n-s_{n-1}),$$

that provides a close approximation to

$$\int\limits_0^{f_\gamma^*(t)}f_{*,\gamma}(s)\varphi_\gamma^{**}(f_{*,\gamma}(s))ds.$$

By adding more points to the Riemann sum if necessary, we may assume that the left-hand end point of each interval on which $f_{*,\gamma}$ is constant is included among the s_n that is contained in the interior of an interval on which $f_{*,\gamma}$ is constant, is deleted. It is now an easy matter to verify that for each of the remaining s_n there is precisely one element, ξ_n, such that $s_n = f_\gamma^*(\xi_n)$ and that $f_{*,\gamma}\left(f_\gamma^*(\xi_n)\right) = \xi_n$. Therefore

$$\sum_{n=1}^{+\infty} f_{*,\gamma}(s_{n-1})\varphi_\gamma^{**}(f_{*,\gamma}(s_{n-1}))(s_n - s_{n-1}) =$$

$$= \sum_{n=1}^{+\infty} \xi_{n-1}\varphi_\gamma^{**}(\xi_{n-1})(f_\gamma^*(\xi_n) - f_\gamma^*(\xi_{n-1})),$$

which, by adding more points if necessary, provides a close approximation to

$$-\int_t^\infty \xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi).$$

Recalling (15) we get

$$(h_2)_\gamma^{**}(t) \le \int_0^{f_\gamma^*(t)} f_{*,\gamma}(s)\varphi_\gamma^{**}(f_{*,\gamma}(s))ds = -\int_t^\infty \xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi). \tag{16}$$

Now let δ be an arbitrarily large number and choose ξ_j such that $t = \xi_1 \le \xi_2 \le \ldots \le \xi_{j+1} = \delta$. Then

$$\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta) - t\varphi_\gamma^{**}(t)f_\gamma^*(t) =$$

$$= \sum_{n=1}^{j} \xi_{n+1}\varphi_\gamma^{**}(\xi_{n+1})\left(f_\gamma^*(\xi_{n+1}) - f_\gamma^*(\xi_n)\right) +$$

$$+ \sum_{n=1}^{j} f_\gamma^*(\xi_n)\left(\varphi_\gamma^{**}(\xi_{n+1})\xi_{n+1} - \varphi_\gamma^{**}(\xi_n)\xi_n\right) =$$

$$= \sum_{n=1}^{j} \xi_{n+1}\varphi_\gamma^{**}(\xi_{n+1})\left(f_\gamma^*(\xi_{n+1}) - f_\gamma^*(\xi_n)\right) +$$

$$+ \sum_{n=1}^{j} f_\gamma^*(\xi_n)\int_{\xi_n}^{\xi_{n+1}} \varphi_\gamma^*(\tau)d\tau \le$$

$$\le \sum_{n=1}^{j} \xi_{n+1}\varphi_\gamma^{**}(\xi_{n+1})\left(f_\gamma^*(\xi_{n+1}) - f_\gamma^*(\xi_n)\right) +$$

$$+ \sum_{n=1}^{j} f_\gamma^*(\xi_n)\varphi^*(\xi_n)\left(\xi_{n+1} - \xi_n\right).$$

This means that

$$\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta) - t\varphi_\gamma^{**}(t)f_\gamma^*(t) \le \int_t^\delta \xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi) + \int_t^\delta f_\gamma^*(\xi)\varphi^*(\xi)d\xi. \tag{17}$$

Now we estimate the expression $\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta) - t\varphi_\gamma^{**}(t)f_\gamma^*(t)$ below.

$$\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta) - t\varphi_\gamma^{**}(t)f_\gamma^*(t) =$$

$$= \sum_{n=1}^{j} \xi_n\varphi_\gamma^{**}(\xi_n)\left(f_\gamma^*(\xi_{n+1}) - f_\gamma^*(\xi_n)\right) +$$

$$+\sum_{n=1}^{j} f_\gamma^*(\xi_{n+1})\left(\varphi_\gamma^{**}(\xi_{n+1})\xi_{n+1}-\varphi_\gamma^{**}(\xi_n)\xi_n\right)=$$

$$=\sum_{n=1}^{j}\xi_n\varphi_\gamma^{**}(\xi_n)\left(f_\gamma^*(\xi_{n+1})-f_\gamma^*(\xi_n)\right)+$$

$$+\sum_{n=1}^{j} f_\gamma^*(\xi_{n+1})\int\limits_{\xi_n}^{\xi_{n+1}}\varphi_\gamma^*(\tau)d\tau\geq$$

$$\geq\sum_{n=1}^{j}\xi_n\varphi_\gamma^{**}(\xi_n)\left(f_\gamma^*(\xi_{n+1})-f_\gamma^*(\xi_n)\right)+$$

$$+\sum_{n=1}^{j} f_\gamma^*(\xi_{n+1})\varphi_\gamma^*(\xi_{n+1})\left(\xi_{n+1}-\xi_n\right).$$

In other words

$$\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta)-t\varphi_\gamma^{**}(t)f_\gamma^*(t)\geq\int\limits_t^\delta\xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi)+\int\limits_t^\delta f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi. \tag{18}$$

From (17) and (18) we obtain

$$-\int\limits_t^\delta\xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi)=t\varphi_\gamma^{**}(t)f_\gamma^*(t)-\delta\varphi_\gamma^{**}(\delta)f_\gamma^*(\delta)+\int\limits_t^\delta f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi\leq$$

$$\leq t\varphi_\gamma^{**}(t)f_\gamma^*(t)+\int\limits_t^\delta f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi.$$

Thus

$$-\int\limits_t^\infty\xi\varphi_\gamma^{**}(\xi)df_\gamma^*(\xi)\leq t\varphi_\gamma^{**}(t)f_\gamma^*(t)+\int\limits_t^\infty f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi.$$

By using this inequality and (16), we have

$$(h_2)_\gamma^{**}(t)\leq\int\limits_0^{f_\gamma^*(t)} f_{*,\gamma}(s)\varphi_\gamma^{**}(f_{*,\gamma}(s))ds\leq t\varphi_\gamma^{**}(t)f_\gamma^*(t)+\int\limits_t^\infty f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi. \tag{19}$$

Finally, from (14), (19) and (6) we get

$$h_\gamma^{**}(t)\leq(h_1)_\gamma^{**}(t)+(h_2)_\gamma^{**}(t)\leq$$

$$\leq\varphi_\gamma^{**}(t)\int\limits_{f_\gamma^*(t)}^\infty f_{*,\gamma}(s)ds+t\varphi_\gamma^{**}(t)f_\gamma^*(t)+\int\limits_t^\infty f_\gamma^*(\xi)\varphi_\gamma^*(\xi)d\xi=$$

$$=f_\gamma^*(t)\varphi_\gamma^{**}(t)+\int\limits_t^\infty f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi=$$

$$=tf_\gamma^{**}(t)\varphi_\gamma^{**}(t)+\int\limits_t^\infty f_\gamma^*(\xi)\varphi_\gamma^*\xi)d\xi.$$

Proof of Theorem 1.4. Assume that the integral on the right of (4) is finite. Then it is easy to see that

$$s f_\gamma^{**}(s)\varphi_\gamma^{**}(s) \to 0, \text{ as } s \to \infty. \tag{20}$$

Let $h = f \otimes \varphi$. By Theorem 1.3 we have

$$h_\gamma^{**}(t) \le t f_\gamma^{**}(t)\varphi_\gamma^{**}(t) + \int_t^\infty f_\gamma^*(s)\varphi_\gamma^*(s)ds \le$$

$$\le t f_\gamma^{**}(t)\varphi_\gamma^{**}(t) + \int_t^\infty f_\gamma^{**}(s)\varphi_\gamma^*(s)ds. \tag{21}$$

Since f_γ^{**} and φ_γ^{**} are non-increasing,

$$\frac{df_\gamma^{**}(s)}{ds} = -\frac{1}{s^2}\int_0^s f_\gamma^*(\tau)d\tau + \frac{1}{s} f_\gamma^*(s) = \frac{1}{s}\left(f_\gamma^*(s) - f_\gamma^{**}(s)\right), \tag{22}$$

$$\frac{d(s\varphi_\gamma^{**}(s))}{ds} = \varphi_\gamma^{**}(s) + s\left(\frac{1}{s}\left(\varphi_\gamma^*(s) - \varphi_\gamma^{**}(s)\right)\right) = \varphi_\gamma^*(s) \tag{23}$$

for m-almost all s. Since f_γ^{**} and φ_γ^{**} are absolutely continuous, we may use the method of the integration by parts for $\int_t^\infty f_\gamma^{**}(s)d\left(s\varphi_\gamma^{**}(s)\right)$. Using (22), (23) and (20) we obtain

$$\int_t^\infty f_\gamma^{**}(s)\varphi_\gamma^*(s)ds = \int_t^\infty f_\gamma^{**}(s)d\left(s\varphi_\gamma^{**}(s)\right) =$$

$$= f_\gamma^{**}(s)s\varphi_\gamma^{**}(s)\big|_t^\infty - \int_t^\infty s\varphi_\gamma^{**}(s)df_\gamma^{**}(s) =$$

$$= -t f_\gamma^{**}(t)\varphi_\gamma^{**}(t) + \int_t^\infty \varphi_\gamma^{**}(s)(f_\gamma^{**}(s) - f_\gamma^*(s))ds \le$$

$$\le -t f_\gamma^{**}(t)\varphi_\gamma^{**}(t) + \int_t^\infty \varphi_\gamma^{**}(s) f_\gamma^{**}(s)ds. \tag{24}$$

By (21) and (24) we have

$$h_\gamma^{**}(t) \le \int_t^\infty f_\gamma^{**}(s)\varphi_\gamma^{**}(s)ds.$$

The proof is completed. The next lemma is a classical estimate known as Hardy's inequality.

Lemma 3.2. *([2]) If $1 \le p < \infty$, $q > 0$ and f is a nonnegative measurable function on $(0,\infty)$, then*

$$\int_0^\infty \left(\frac{1}{s}\int_0^s f(\tau)d\tau\right)^p s^{p-q-1}ds \le \left(\frac{p}{q}\right)^q \int_0^\infty f(t)^p t^{p-q-1}dt. \tag{25}$$

Proof of Theorem 1.5. Let $h = f \otimes \varphi$.
Suppose that q_1, q_2, q_0 are finite numbers. Then, by (4) we have

$$(\|h\|_{p_0,q_0,\gamma})^{q_0} = \int_0^\infty \left(s^{\frac{1}{p_0}} h_\gamma^{**}(s)\right)^{q_0} \frac{ds}{s} \le$$

$$\leq \int_0^\infty \left(s^{\frac{1}{p_0}} \int_s^\infty f_\gamma^{**}(\tau)\varphi_\gamma^{**}(\tau)d\tau \right)^{q_0} \frac{ds}{s} =$$

$$= \int_0^\infty \left(\frac{1}{t^{\frac{1}{p_0}}} \int_0^t f_\gamma^{**}\left(\frac{1}{\eta}\right) \varphi_\gamma^{**}\left(\frac{1}{\eta}\right) \frac{d\eta}{\eta^2} \right)^{q_0} \frac{dt}{t}.$$

The last equality was obtained by the change of variables $s = \dfrac{1}{t}$ and $\tau = \dfrac{1}{\eta}$. Using (25), we get

$$\int_0^\infty \left(\frac{1}{t^{\frac{1}{p_0}}} \int_0^t f_\gamma^{**}\left(\frac{1}{\eta}\right) \varphi_\gamma^{**}\left(\frac{1}{\eta}\right) \frac{d\eta}{\eta^2} \right)^{q_0} \frac{dt}{t} \leq$$

$$\leq p_0^{q_0} \int_0^\infty \left(t^{1-\frac{1}{p_0}} \frac{f_\gamma^{**}\left(\frac{1}{t}\right) \varphi_\gamma^{**}\left(\frac{1}{t}\right)}{t^2} \right)^{q_0} \frac{dt}{t} =$$

$$= p_0^{q_0} \int_0^\infty \left(s^{1+\frac{1}{p_0}} f_\gamma^{**}(s)\, \varphi_\gamma^{**}(s) \right)^{q_0} \frac{ds}{s}$$

The last equality was obtained by the change of the variable $t = \dfrac{1}{s}$. Since $\dfrac{q_0}{q_1} + \dfrac{q_0}{q_2} \geq 1$, one can find positive numbers n_1 and n_2 such that

$$\frac{1}{n_1} + \frac{1}{n_2} = 1 \text{ and } \frac{1}{n_1} \leq \frac{q_0}{q_1},\ \frac{1}{n_2} \leq \frac{q_0}{q_2}.$$

By Hölder's inequality we obtain

$$(\|h\|_{p_0,q_0,\gamma})^{q_0} \leq p_0^{q_0} \int_0^\infty \frac{\left(s^{\frac{1}{p_1}} f_\gamma^{**}(s)\right)^{q_0}}{s^{\frac{1}{n_2}}} \frac{\left(s^{\frac{1}{p_2}} \varphi_\gamma^{**}(s)\right)^{q_0}}{s^{\frac{1}{n_1}}} ds \leq$$

$$\leq p_0^{q_0} \left[\int_0^\infty \left(s^{\frac{1}{p_1}} f_\gamma^{**}(s)\right)^{q_0 n_1} \frac{ds}{s} \right]^{\frac{1}{n_1}} \left[\int_0^\infty \left(s^{\frac{1}{p_2}} \varphi_\gamma^{**}(s)\right)^{q_0 n_2} \frac{ds}{s} \right]^{\frac{1}{n_2}} =$$

$$= p_0^{q_0} \left(\|f\|_{p_1,q_0 n_1,\gamma}\right)^{q_0} \left(\|\varphi\|_{p_2,q_0 n_2,\gamma}\right)^{q_0}.$$

Finally, by (8) we have

$$\|h\|_{p_0,q_0,\gamma} \leq p_0 \|f\|_{p_1,q_0 n_1,\gamma} \|\varphi\|_{p_2,q_0 n_2,\gamma} \leq p_0 e^{\frac{1}{e}} e^{\frac{1}{e}} \|f\|_{p_1,q_1,\gamma} \|\varphi\|_{p_2,q_2,\gamma} \leq 3p_0 \|f\|_{p_1,q_1,\gamma} \|\varphi\|_{p_2,q_2,\gamma}.$$

Similar reasoning leads to the desired result in case one or more of q_1, q_2, q_0 are ∞. Theorem 1.5 is proved.

4. One particular case of theorem 1.5

Consider the following particular case of Theorem 1.5. If we take $p_1 = \dfrac{n+|\gamma|}{n+|\gamma|-\alpha}$ with $\alpha > 0$, and $q_1 = \infty$ in Theorem 1.5, then the condition $\dfrac{1}{p_1} + \dfrac{1}{p_2} > 1$ is equivalent to $\alpha < \dfrac{n+|\gamma|}{p_2}$, and the condition $\dfrac{1}{p_1} + \dfrac{1}{p_2} - 1 = \dfrac{1}{q_0}$ is equivalent to $\dfrac{1}{q_0} = \dfrac{1}{p_2} - \dfrac{\alpha}{n+|\gamma|}$. Thus we have the following results.

Corollary 4.1. *If* $f \in L_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\varphi \in L_{p,q,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *where* $0<\alpha<\dfrac{n+|\gamma|}{p}$, *then* $(f\otimes\varphi)\in L_{r,q,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|(f\otimes\varphi)\|_{r,q,\gamma}\le 3r\|f\|_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\|\varphi\|_{p,q,\gamma}, \tag{26}$$

where $\dfrac{1}{r}=\dfrac{1}{p}-\dfrac{\alpha}{n+|\gamma|}$.

Corollary 4.2. *If* $f \in L_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\varphi \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *where* $0<\alpha<\dfrac{n+|\gamma|}{p}$, *then* $(f\otimes\varphi)\in L_{r,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|(f\otimes\varphi)\|_{r,\gamma}\le 3r\frac{p}{p-1}\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}}\|f\|_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\|\varphi\|_{p,\gamma}. \tag{27}$$

where $\dfrac{1}{r}=\dfrac{1}{p}-\dfrac{\alpha}{n+|\gamma|}$.

Proof. From (7), (8) and (26) we have

$$\|(f\otimes\varphi)\|_{r,\gamma}\le\|(f\otimes\varphi)\|_{r,r,\gamma}\le\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}}\|(f\otimes\varphi)\|_{r,p,\gamma}$$

$$\le 3r\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}}\|f\|_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\|\varphi\|_{p,p,\gamma}\le 3r\frac{p}{p-1}\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}}\|f\|_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\|\varphi\|_{p,\gamma}.$$

Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n_{k,+}$, i.e., $\Omega(sx)=\Omega(x)$ for all $s>0$, $x\in\mathbb{R}^n_{k,+}$, and $\Omega\in L_{\frac{n+|\gamma|}{n+|\gamma|-\alpha}}\left(S^{n-1}_{k,+}\right)$, where $0<\alpha<n+|\gamma|$, and $S^{n-1}_{k,+}=\{x\in\mathbb{R}^n_{k,+}:|x|=1\}$. Define the B-fractional integral (or Riesz potential) by

$$I_{\Omega,\alpha,\gamma}f(x)=\int\limits_{\mathbb{R}^n_{k,+}}\frac{\Omega(y)}{|y|^{n+|\gamma|-\alpha}}\left(T^x f(y)\right)\left(y'\right)^{\gamma}dy.$$

It is easily checked that $\dfrac{\Omega(y)}{|y|^{n+|\gamma|-\alpha}}\in L_{\frac{n+|\gamma|}{n+|\gamma|-\alpha},\infty,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, if $0<\alpha<\dfrac{n+|\gamma|}{p}$. Then one can get by Corollary 4.1 and Corollary 4.2 the Hardy-Littlewood-Sobolev theorem for the B-fractional integrals on Lorentz and Lebesgue spaces (see [4]), respectively.

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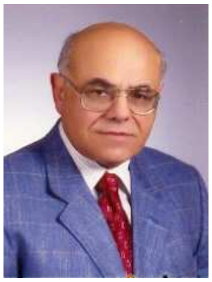

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